# Toroidal electron temperature gradient mode structure in the presence of nonuniform background flows

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The influence of nonuniform poloidal and toroidal background plasma flows on the spatial structure and growth rate of the electrostatic electron temperature gradient (ETG) mode is investigated in the linear approximation. This derivation includes the ballooning mode formalism and a more recently developed version of the direct method by Taylor and Wilson [Plasma Phys. Controlled Fusion **38**, 1999 (1996)]. It is shown that the growth rate of the ETG mode is not changed significantly by flow shear. However, it is found that the spatial structure of the ETG mode depends crucially on the derivative of the flow shear rate with respect to the minor radius of the tokamak cross section and also depends crucially on the magnetic shear. For moderate magnetic shear, the unstable ETG mode is strongly localized in the poloidal direction and is elongated along the radial direction, with a characteristic radial width much larger than the electron Larmor radius. This may explain the formation of streamer structures above the threshold of ETG mode instability. Streamers are believed to enhance electron thermal transport beyond the values provided by simple mixing length estimates. For very low values of magnetic shear, the ETG mode structure becomes extended in the poloidal direction, and the ballooning formalism does not apply. In this case, the direct method is used and it is shown that the ETG mode is strongly localized in the radial direction. The small radial extent of these modes may considerably reduce electron heat transport, which would enhance the formation of an electron thermal transport barrier. 
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## I. INTRODUCTION

Many experimental observations in tokamaks (see Refs. 1–4, and references therein) as well as numerical simulations[5–9] show that the electron thermal transport is anomalous inside internal transport barriers (ITBs), which are formed in regions of strong plasma rotation flow shear. The region of reduced ion thermal transport inside an ITB coincides approximately with the region where the flow shear rate, $\omega_E$, exceeds the linear growth of the most unstable ion temperature gradient (ITG) mode, $\gamma_{L\,\max}$,

$$\omega_E \equiv \left(\frac{r}{q}\right)\frac{d}{dr}\left(\frac{qc\mathbf{E}\times\mathbf{B}}{rB^2}\right) > \gamma_{L\,\max}, \tag{1}$$

where $r$ is the minor radius (half-width of the flux surface), $q$ is the safety factor as a function of the flux surface, $\mathbf{E}$ is the radial electric field, and $\mathbf{B}$ is the magnetic field strength (typically measured at the outboard edge of each flux surface).

The influence of nonuniform plasma rotation on ITG modes in tokamaks was considered in Ref. 10 using the direct method (DM),[11] which was developed for dissipative electron drift waves in plasmas with a background velocity profile that is a linear function of the minor radius. This method was later generalized to the case of a parabolic velocity profile in Ref. 12. The effective stabilization given by Eq. (1) has been qualitatively confirmed for reactive ITG modes described by an advanced fluid model.[13–15] Anomalous electron heat transport is now believed to be caused by the most unstable electron temperature gradient (ETG) mode or by a trapped electron mode.

The stability properties and spatial structure of the ETG mode will be studied in this paper. In the absence of plasma rotation, the linear theory of the ETG mode instability is well established.[16–19] The similarity between ITG and ETG modes was pointed out [with the role of electrons and ions exchanged and the maximum growth rate for the ETG mode larger than the corresponding ITG mode by a factor $(m_i/m_e)^{1/2}$, where $m_i$ is the ion mass and $m_e$ is the electron mass]. Thus, when the Eq. (1) holds for the ITG mode and the ITG mode is suppressed, the ETG mode is not necessarily suppressed. Also, in the absence of plasma rotation, the characteristic width of the ETG mode is smaller than the width of the ITG mode counterpart by a factor on the order of $(m_e/m_i)^{1/2}$. As a result, the ETG mode width is much shorter than the characteristic scale length of the sheared background flow. Because of this, it is now widely believed that plasma rotation has only a slight influence on the ETG mode.

The simple mixing rate estimate for the electron heat conductivity, $\chi_e$, which is qualitatively right for the ITG mode, indicates that the transport driven by ETG modes would be smaller by a factor $(m_e/m_i)^{1/2}$ than the transport driven by ITG modes. The last conclusion is not consistent

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with experimental observations, at least inside ITBs,[1] where $\chi_e$ is observed to be much larger than the ion thermal conductivity, $\chi_i$, and where $\chi_e$ remains far above the neoclassical level. This inconsistency may be removed by considering the results of gyrokinetic simulations,[9] which indicate that radially elongated structures, called "streamers," appear just after the electron temperature gradient exceeds the critical instability value.

It is believed that an appearance of streamers is caused by nonlinear effects, which lead to a merging of smaller structures. It is shown in this paper, however, that under appropriate conditions, radially elongated structures can appear in the linear stage of ETG modes. Hence, radially elongated structures are not exclusively a nonlinear phenomenon.

It has been shown in Refs. 20–22 that the radial nonuniformity of plasma parameters such as the diamagnetic frequency have a strong influence on the spatial structure and growth rate during the linear stage of dissipative electron drift and ITG modes. It is assumed that strong flow shear in regions of low magnetic shear also has a strong effect on the radial structure and growth rate of ITG modes.[23,24] The reduction in transport caused by the presence of velocity flow shear in regions of low magnetic shear is consistent with integrated modeling simulations of internal transport barriers in tokamaks using models for transport driven by ITG and trapped electron modes.[7]

The primary objective of the present work is to study the linear ETG mode spatial structure and, in particular, radially elongated ETG mode structures in the presence of nonuniform toroidal and poloidal plasma flows, using both the strong ballooning approximation (SBA) and direct method (DM).

This paper is organized as follows. In Sec. II, the theoretical model and the basic two-dimensional (2D) differential equation, which describes the ETG eigenmode structures in a tokamak cross section, are presented. In Sec. III, this structure is analyzed using the strong ballooning formalism (SBF) (Refs. 20, 25) up to the second order approximation. In Sec. IV, the DM is used in carrying out for a more detailed study of spatial structure of the ETG modes, in the cases of both low and moderate magnetic shears. Finally, in Sec. V, the influence of the ETG mode structure on the electron heat transport is discussed. The influence of nonuniform background flow on electron heat transport appears to result from a modification of the ETG mode structure in radial direction and the formation of "streamers" during the linear stage of the instability.

## II. BASIC EQUATIONS

An electrostatic model for ETG modes, as described in Refs. 17 and 18, is used and toroidal and poloidal rotation in the tokamak plasma are taken into account. Also, the effects of impurities, superthermal ions and noncircular flux surfaces are considered. Gyrokinetic simulations[19] have confirmed that electromagnetic effects have very little influence on the ETG mode linear instability (see, for example, Fig. 1 of Ref. 19). In the work of Singh *et al.*,[18] it was shown that the advanced fluid model developed in Refs. 13–15 yields results that are qualitatively similar to the kinetic approach, especially for plasma conditions above the ETG instability threshold $\eta_{e,\mathrm{cr}}$ [$\eta_e \gg \eta_{e,\mathrm{cr}}$, where $\eta_e \equiv (d\ln T_e)/(d\ln n_e)$, $T_e$ is the electron temperature, and $n_e$ is the electron density].

This advanced fluid model will be used in this paper with the assumption that the ETG mode is electrostatic and that ions can be treated adiabatically. Note that deviations from ion adiabaticity were investigated by Singh *et al.*[18] These deviations were found to produce a very small stabilizing effect on the ETG instability. In addition, Debye shielding effects were shown in Ref. 18 to have only a small influence on the growth rate, at least for $\lambda_{\mathrm{De}}^2/\rho_e^2 < 1$, which is typical for modern tokamaks (where $\lambda_{\mathrm{De}}$ is the electron Debye length and $\rho_e$ is the electron Larmor radius).

For ETG modes, typical space ($k_\perp^{-1}$) and temporal ($\omega^{-1}$) scales are such that[17,18]

$$\rho_e^{-1} \gg k_\perp \gg \rho_i^{-1},$$

$$k_\parallel c_e < \omega \ll k_\perp c_i,$$

and

$$\max(\omega,\gamma) \ll \Omega_e,$$

where $\rho_i$ is the ion Larmor radius, $c_e$ is the electron thermal velocity, $c_i$ is the thermal velocity of the major ion species, $\Omega_e$ is the electron Larmor frequency, $k_\perp$ and $k_\parallel$ are the components of the wave number perpendicular and parallel to the background magnetic field, $\omega$ is the real part of the ETG mode frequency, and $\gamma$ is the growth rate of the unstable mode.

Consider multiple species of ions with density $n_j$ and temperature $T_j$, where $j=H$, for hydrogenic ions, $j=Z$, for impurity ions with charge $Z$, and $j=S$, for superthermal ions with charge $Z_S$ (such as fast alpha particles or neutral beam injection ions). Each density is divided into a background density, $n_{0j}$, and a perturbed density $\delta n_j$. If the perturbed ion densities are adiabatic, the following relation holds

$$\frac{\delta n_j}{n_{0j}} = -\tau_j \Phi, \tag{2}$$

where $\tau_j \equiv T_e/T_j$, $\Phi \equiv -e\phi/T_e$, and $\phi$ is the electric potential of the perturbation. If $f_j \equiv n_{0j}/n_{0e}$, for $j=H$, $Z$, or $S$, where $n_0 \equiv n_{0e}$ is the background electron density, the quasineutrality condition can be written $n_{0e} = n_{0H} + Zn_{0Z} + Z_S n_{0S}$ or, equivalently,

$$\frac{n_{0H}}{n_{0e}} = 1 - Zf_Z - Z_S f_S.$$

The Poisson equation for the perturbed electric potential may be written in the form

$$\lambda_{\mathrm{De}}^2 \nabla_\perp^2 \Phi = \frac{\delta n_e}{n_{0e}} + \Phi[\tau_H(1-Zf_Z-Z_Sf_S) + Zf_Z\tau_Z + Z_Sf_S\tau_S]$$

$$= \frac{\delta n_e}{n_{0e}} + \Phi\tau, \tag{3}$$

where

$$\tau \equiv (1-Zf_Z-Z_Sf_S)\tau_H + Zf_Z\tau_Z + Z_Sf_S\tau_S.$$

The continuity equation for the perturbed electron density takes the form

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla\right)\delta n_e+\nabla_\perp(n_{0e}\mathbf{v}_e)+\nabla_\parallel(n_{0e}\mathbf{v}_{e\parallel})=0, \tag{4}$$

where $\mathbf{v}_{e\parallel}$ is the electron velocity perturbation parallel to the magnetic field $\mathbf{B}$, which satisfies the equation,

$$m_e\left[\left(\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla\right)v_{e\parallel}+v_{er}\frac{d}{dr}v_{0\parallel}\right]$$
$$=\nabla_\parallel(\Phi-\delta T_e/T_e-\delta n_e/n_{0e}), \tag{5}$$

where $v_{0\parallel}$ is background parallel electron velocity and $v_{er}$ is projection of perturbated electron velocity on the radial direction.

In the electron continuity equation, the electron drift across the magnetic field $\mathbf{B}$, assuming $\omega\ll\Omega_e$, is

$$\mathbf{v}_{\perp e}=\mathbf{v}_E+\mathbf{v}_{e*}+\mathbf{v}_{ep}+\mathbf{v}_{e\Pi},$$

where $\mathbf{v}_E$ is the $\mathbf{E}\times\mathbf{B}$ drift,

$$\mathbf{v}_E=\frac{c}{B_0^2}\mathbf{E}\times\mathbf{B},$$

$\mathbf{v}_{*e}$ is the diamagnetic drift,

$$\mathbf{v}_{*e}=-\frac{c}{en_{0e}B_0^2}\mathbf{B}\times\nabla p_e,$$

$\mathbf{v}_{\rm De}$ is the drift due to $\nabla|\mathbf{B}|$ and magnetic curvature,

$$\mathbf{v}_{\rm De}=\frac{T_e}{m_e\Omega_e}\mathbf{e}_\parallel\times[(\mathbf{e}_\parallel\cdot\nabla)\mathbf{e}_\parallel+\nabla|\mathbf{B}|/B_0].$$

$\mathbf{v}_{ep}$ is the polarization drift,

$$\mathbf{v}_{ep}=\frac{d\mathbf{E}}{dt}/(B_0\Omega_e),$$

where

$$\frac{d}{dt}\mathbf{E}=\left(\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla\right)\mathbf{E},\quad \mathbf{E}=-\nabla\phi,$$

and $\mathbf{v}_{e\Pi}$ is the drift due to the off-diagonal elements of the stress tensor $\Pi_e$,

$$\mathbf{v}_{e\Pi}=-\frac{c\mathbf{B}\times\nabla\Pi_e}{en_{0e}B_0^2}.$$

The equation for the evolution of the electron temperature perturbation may be written

$$\frac{3}{2}n_0\left(\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla\right)\delta T_e+\nabla\cdot\left(\frac{3}{2}n_0T_e\mathbf{v}_{\perp e}+\mathbf{q}_{*e}\right)$$
$$+n_0T_e\nabla\cdot\mathbf{v}_{\perp e}=0, \tag{6}$$

where $\mathbf{q}_{*e}$ is the electron heat flow,

$$\mathbf{q}_{*e}=\frac{5}{2}\frac{p_e}{m_e\Omega_e}(\mathbf{e}_\parallel\times\nabla)T_e,\quad n_0\equiv n_{0e},$$

$p_e=n_{0e}T_{0e}$ is the background electron pressure, $\mathbf{v}_0$ is the background electron flow velocity, $v_{0\parallel}$ is the toroidal component of the flow velocity. The flow velocity is assumed to be a function of the minor radius $\mathbf{v}_0=\mathbf{v}_0(r)$.

The system of equations, Eqs. (3)–(6), is a linearized set of equations that determine the spatial-temporal evolution of the ETG instability in axisymmetric tokamaks with circular cross section. The generalization for tokamaks with elongated cross section will be discussed in the final section of this paper.

Following the procedure described, for example, in Ref. 14, Eqs. (4)–(6) can be written in the form

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla+\mathbf{v}_{\rm De}\cdot\nabla\right)\frac{\delta n_e}{n_0}+(\mathbf{v}_{*e}-\mathbf{v}_{\rm De})\cdot\nabla\Phi$$
$$+\mathbf{v}_{\rm De}\cdot\nabla\delta T_e+\frac{1}{n_0}\nabla\cdot(n_{0e}\mathbf{v}_{\parallel e})$$
$$+\rho_e^2\left[\frac{\partial}{\partial t}+\mathbf{v}_{*e}(1+\eta_e)\cdot\nabla\right]\nabla^2\Phi=0, \tag{7}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla\right)v_{\parallel e}+v_{re}\frac{d}{dr}v_{0\parallel}=c_e^2\left(\Phi-\frac{\delta T_e}{T_e}-\frac{\delta n_e}{n_0}\right), \tag{8}$$

$$\left(\frac{\partial}{\partial t}+\frac{5}{3}\mathbf{v}_{\rm De}\cdot\nabla+\mathbf{v}_0\cdot\nabla\right)\frac{\delta T_e}{T_e}+(\eta_e-2/3)\mathbf{v}_{*e}\cdot\nabla\Phi$$
$$-\frac{2}{3}\left(\frac{\partial}{\partial t}+\mathbf{v}_0\cdot\nabla\right)\frac{\delta n_e}{n_0}=0. \tag{9}$$

It is assumed that all of the perturbed variables $\phi$, $\delta n_e$, $\delta T_e$, and $\delta v_{\parallel e}$ have the form

$$f(r,\theta,\zeta)\exp(im\theta-in\zeta-i\omega t),$$

where $\theta$ is the poloidal angle with mode number $m$, and $\zeta$ is the toroidal angle with toroidal mode number $n\gg1$. The functional form $f(r,\theta,\zeta)$ for each perturbed variable is assumed to vary slowly with $\zeta$ compared with the exponential dependence $\exp(in\zeta)$.

Simulations of DIII-D,[1,4] Tokamak Fusion Test Reactor (TFTR),[2] and other tokamaks have shown that the flow shear rate

$$\omega_E=\frac{r}{q}\frac{d}{dr}\left(\frac{q}{r}\frac{cE_r}{B_0}\right)$$

has a strong radial gradient, $d\omega_E/dr$, inside internal transport barriers. Note that $\omega_E$ can be approximated with

$$\frac{d}{dr}\left(\frac{cE_r}{B_0}\right)\approx\frac{d}{dr}v_{0\theta}$$

in regions with sharp gradients of electric field, where $v_{0\theta}$ is the poloidal component of the background electron velocity.

The background velocity flow is approximated using a parabolic function

$$\mathbf{v}_0(r)=\mathbf{v}_0(r_0)+\mathbf{v}_0'(r_0)\rho+(1/2)\mathbf{v}_0''(r_0)\rho^2,$$

where $\rho\equiv r-r_0$, $r_0$ is the radius of the mode rational surface, and $nq(r_0)=m$. Terms with the form $(\mathbf{v}_0\cdot\nabla)f$ on the left-hand side of Eqs. (4)–(9) result in a Doppler shift of the ETG perturbation frequency $\omega$:

$$\omega\rightarrow\omega-k_\theta\left[v_{0\theta}|_{r=r_0}+\frac{dv_{0\theta}}{dr}\bigg|_{r=r_0}\rho+\frac{1}{2}\frac{d^2v_{0\theta}}{dr^2}\bigg|_{r=r_0}\rho^2\right], \tag{10}$$

where $k_\theta=m/r_0$. The contribution from the toroidal component of the background velocity to the Doppler shift is negligible because $|k_\theta|\gg|k_\parallel|$. However, the contribution of toroidal flow velocity in the last term on the LHS of Eq. (8) is kept and it is approximated as follows:

$$v_r\frac{d}{dr}v_{0\parallel}\sim-\frac{c}{B}\nabla_\perp\phi\frac{d}{dr}v_{0\parallel}\sim-\frac{ic_e^2k_\theta}{\Omega_e}\Phi\frac{d}{dr}v_{0\parallel}.$$

From Eqs. (7) to (9), in a framework moving with constant velocity $v_{0\theta}$, one obtains a 2D differential equation for the function $G\equiv\Phi-\delta T_e/T_e$,

$$\Bigg\{\left[1+\frac{k_\theta^2(\lambda_{\mathrm{De}}^2+\rho_e^2)}{\tau}\right]\omega^2+\left[\frac{\omega_{*e}}{\tau}-\omega_{\mathrm{De}}\left(\frac{10}{3}+\frac{1}{\tau}\right)-\underbrace{\frac{10}{3}\frac{\omega_{\mathrm{De}}\lambda_{\mathrm{De}}^2\hat{k}_\perp^2}{\tau}}-\left[\omega_{*e}(1+\eta_e)+\frac{5\omega_{\mathrm{De}}}{3}\right]\frac{\hat{k}_\perp^2\rho_e^2}{\tau}\right]\omega$$
$$+\frac{5}{3}\omega_{\mathrm{De}}^2\left(1+\underbrace{\frac{\hat{k}_\perp^2\lambda_{\mathrm{De}}^2}{\tau}}\right)+\frac{5}{3\tau}\omega_{\mathrm{De}}^2+\frac{1}{\tau}\left(\eta_e-\frac{7}{3}\right)\omega_{\mathrm{De}}\omega_{*e}+\frac{5}{3\tau}(1+\eta_e)\omega_{\mathrm{De}}\omega_{*e}\hat{k}_\perp^2\rho_e^2\Bigg\}G$$
$$=\left[\omega\left(\frac{5}{3}+\frac{1}{\tau}\right)-\frac{5}{3}\omega_{\mathrm{De}}\left(1+\frac{1}{\tau}\right)-\frac{\omega_{*e}}{\tau}\left(\eta_e-\frac{2}{3}\right)+\underbrace{\frac{5}{3}\hat{k}_\perp^2\lambda_{\mathrm{De}}^2(\omega-\omega_{\mathrm{De}})}\right]\frac{\hat{k}_\parallel^2c_e^2}{\omega}G+\frac{\hat{k}_\parallel k_\theta c_e^2}{\omega\tau\Omega_e}\left(\frac{d}{dr}v_{0\parallel}\right)\Phi \tag{11}$$

together with one algebraic equation connecting $G$ and $\Phi$,

$$\Phi=\frac{(\omega-5\omega_{\mathrm{De}}/3)G}{(1+\tau+k_\theta^2\lambda_{\mathrm{De}}^2)(\omega-5\omega_{\mathrm{De}}/3)+2\tau\omega/3-\omega_{*e}(\eta_e-2/3)}. \tag{12}$$

The operators $\hat{k}_\parallel$ and $\hat{k}_\perp$ introduced in Eq. (11) are defined as follows:

$$\hat{k}_\parallel=-\frac{i}{qR}\left(\frac{\partial}{\partial\theta}+iq'n\rho\right)=-\frac{i}{qR}\left(\frac{\partial}{\partial\theta}+iy\right)$$

and

$$\hat{k}_\perp^2=k_\theta^2\left(1-s^2\frac{\partial^2}{\partial y^2}\right),$$

where

$$y\equiv nq'\rho=k_\theta s\rho \tag{13}$$

and $s\equiv r_0q'/q$ is the magnetic shear.

To take into account nonuniform poloidal rotation, one should replace $\omega$ with the Doppler shifted frequency

$$\omega-k_\theta\frac{dv_{0\theta}}{d\rho}\bigg|_{\rho=0}\rho-\frac{k_\theta}{2}\frac{d^2v_{0\theta}}{d\rho^2}\bigg|_{\rho=0}\rho^2$$

in Eqs. (11) and (12). As in Refs. 17 and 18, where it is assumed that $k_\perp^2\lambda_{\mathrm{De}}^2\ll1$ and $\lambda_{\mathrm{De}}^2\ll\rho_e^2$, the terms in Eq. (11) that are denoted with underbraces can be neglected.

In the last term on the right-hand side of Eq. (11), the fact that $|\Phi|\approx|G|$ can be used and the equation can be simplified by replacing $\Phi$ by $G$. Then using the dimensionless variables

$$\Omega\equiv\frac{\omega R}{c_e},$$

$$k\equiv k_\theta\rho_e,$$

Eq. (11) can be rewritten in dimensionless form

$$\left\{A\left(\frac{\partial}{\partial\theta}+iy\right)^2-B+C\frac{\partial^2}{\partial y^2}+iD\left(\frac{\partial}{\partial\theta}+iy\right)\right\}G=0, \tag{14}$$

where

$$A=\frac{k}{\Omega q^2}\left[\frac{1}{\tau}\left(g_{Te}-\frac{2g_{ne}}{3}\right)+\frac{10}{3}\left(1+\frac{1}{\tau}\right)g_{ne}-\frac{\Omega}{k}\left(\frac{1}{\tau}+\frac{5}{3}\right)\right], \tag{15}$$

$$B=\Omega^2\left(1+\frac{k^2}{\tau}\left(1+\frac{\Omega_e^2}{\omega_{pe}^2}\right)\right)+\Omega k\left[g_{ne}-2\left(\frac{1}{\tau}+\frac{10}{3}\right)g(\theta)-\frac{k^2}{\tau}\left(g_{ne}+g_{Te}+\frac{10}{3}g(\theta)\right)\right]+\frac{2k^2g(\theta)}{\tau}\left[g_{Te}-\frac{7}{3}g_{ne}+\frac{10}{3}(1+\tau)g(\theta)+\frac{10}{3}k^2g(\theta)(g_{ne}+g_{Te})\right],$$

$$C=\frac{k^2s^2}{\tau}\left[\frac{\Omega^2}{\tau}\left(1+\frac{\Omega_e^2}{\omega_{pe}^2}\right)-\Omega k\left(g_{ne}+g_{Te}+\frac{10}{3}\right)+\frac{10}{3}k^2(g_{ne}+g_{Te})\right], \tag{16}$$

$$D=\frac{ks}{qc_e}\frac{1}{\Omega}\frac{dv_{0\parallel}}{dy}, \tag{17}$$

where use has been made of the following notations:

$$\omega_{\mathrm{De}}\equiv\frac{2k_\theta T_e c}{eBR}g(\theta),$$

$$\omega_{*e}\equiv\frac{k_\theta c T_e}{eB}\frac{g_{ne}}{R},$$

$$\eta_e\equiv g_{Te}/g_{ne},$$

$$g_{Te}\equiv-\frac{R}{T_e}\frac{dT_e}{dr},$$

$$g_{ne}\equiv-\frac{R}{n_e}\frac{dn_e}{dr},$$

$$g(\theta)\equiv\cos\theta+is(\sin\theta)\frac{\partial}{\partial y}-\alpha_m\sin^2\theta, \tag{18}$$

and $\alpha_m=-q^2Rd\beta/dr$ is the Shafranov shift term resulting from finite $\beta$ effects in the equilibrium.

When $D=0$, $(\partial/\partial\theta+iy)\rightarrow\partial/\partial\theta$, and $C\partial^2/\partial y^2\rightarrow -C\theta^2$, which is typical for the ballooning formalism, Eq. (14) coincides with Eq. (17) in Ref. 17 (which is written using a different dimensionless variable notation). Also, when $\omega^2\ll k_\perp^2 c_i^2$ and $k_\perp^2\lambda_{\mathrm{De}}^2\ll 1$, Eq. (14) coincides with Eq. (41) of Ref. 18 if $g(\theta)=1$ and if $\tau^*$ and $\epsilon_n$ in Ref. 18 is identified with $\tau$ and $2/g_{ne}$ in this paper.

In order to take into account the poloidal plasma rotation, one should replace $\Omega$ by

$$\Omega-2Ky-Py^2,$$

where

$$K\equiv\frac{\omega_E}{2s}\frac{R}{c_e}, \tag{19}$$

$$P\equiv\frac{R}{2ks^2\Omega_e}\frac{d\omega_E}{d\rho}, \tag{20}$$

and

$$\omega_E=\frac{dv_{0\theta}}{d\rho}.$$

In the next section, Eq. (14) is analyzed in the strong ballooning approximation (SBA).

## III. STRUCTURE OF ETG MODES IN THE STRONG BALLOONING APPROXIMATION

In the ballooning formalism (see Ref. 25, for example), the function $G(y,\theta)$ is represented by the expression

$$G(y,\theta)=\sum_m e^{-im\theta}\int_{-\infty}^{\infty}d\eta e^{im\eta}\overline{G}(y,\eta), \tag{21}$$

where

$$\overline{G}(y,\eta)=G_0(y)f(\eta)e^{-iy(\eta-\eta_k)}.$$

In this expression, $G_0(y)$ is an envelope, which will be obtained in the second order of the SBA.

The first order of the SBA produces a one dimensional eigenvalue problem

$$\left\{A\frac{\partial^2}{\partial\eta^2}+iD\frac{\partial}{\partial\eta}-B_1-C(\eta-\eta_k)^2 +E[\cos\eta-\alpha_m\sin^2\eta+s(\eta-\eta_k)\sin\eta]\right\}f(\eta)=0. \tag{22}$$

In this expression,

$$E=2k\Omega\left[\frac{1}{\tau}+\frac{10}{3}+\frac{10}{3\tau}k^2\right] -\frac{2k^2}{\tau}\left[g_{Te}-\frac{7}{3}g_{ne}+\frac{10}{3}k^2(g_{ne}+g_{Te})\right], \tag{23}$$

and

$$B_1=\Omega^2(1+k^2d_e^2)+\Omega k[g_{ne}-k^2(g_{ne}+g_{Te})/\tau], \tag{24}$$

where

$$d_e^2=\frac{1}{\tau}\left(1+\frac{\Omega_e^2}{\omega_{pe}^2}\right).$$

Note that $B_1-E=B(g(\theta=0))\equiv B_0$.

By using the following transformation,

$$f(\eta)=\exp\left(-i\frac{D}{2A}\eta\right)f_1(\eta),$$

Eq. (22) reduces to

$$\left\{A\frac{\partial^2}{\partial\eta^2}+\frac{D^2}{4A}-B_1-C(\eta-\eta_k)^2 +E[\cos\eta-\alpha_m\sin^2\eta+s(\eta-\eta_k)\sin\eta]\right\}f_1(\eta)=0.$$

In the strong ballooning limit, it is assumed that the function $f_1(\eta)$ is strongly localized near the value $\eta=\eta_k$, where the effective potential is minimum. Expansion of the trigonometric functions near $\eta=\eta_k$, up to second order in $(\eta-\eta_k)^2$, yields

$$\left\{A\frac{\partial^2}{\partial\eta^2}-\left(B_1-\frac{D^2}{4A}\right)+E[\cos\eta_k-\alpha_m\sin^2\eta_k] -C_1\left(\eta-\eta_k-\frac{E_1}{2C_1}\right)^2+\frac{E_1^2}{4C_1}\right\}f_1=0, \tag{25}$$

where

$$E_1=E[(1-s)\sin\eta_k-\alpha_m\sin(2\eta_k)] \tag{26}$$

and

$$C_1=C+E(1/2-s)\cos\eta_k+E\alpha_m\cos^2\eta_k. \tag{27}$$

The effective potential in Eq. (25) has extremum at points that satisfy the equation

$$\eta_k=-E_1/(2C_1). \tag{28}$$

One of the most important solutions of Eq. (28) is $\eta_k=0$, on the outboard edge of the torus, where $E_1=0$ and the potential usually has the deepest minimum.

### A. The first order strong ballooning approximation

In the first order SBA, substituting $\eta_k = 0$ in Eq. (25), yields

$$\left\{ A \frac{\partial^2}{\partial \eta^2} - \left( B_1 - \frac{D^2}{4A} \right) + E - \left[ C + E\left( \frac{1}{2} - s \right) \right] \eta^2 \right\} f_1 = 0. \tag{29}$$

Let $B_2 = B_1 - E - D^2/(4A)$ and $C_2 = C + E(1/2 - s)$, so that Eq. (29) can be rewritten as

$$\left\{ A \frac{\partial^2}{\partial \eta^2} - B_2 - C_2 \eta^2 \right\} f_1 = 0. \tag{30}$$

Equation (30) coincides with Eq. (41) of Ref. 18 after the corresponding renormalization.

A particular physically relevant solution of Eq. (30) is given by

$$f_1 = f_0 \exp(-\sigma \eta^2/2), \tag{31}$$

where

$$\sigma = \sqrt{C_2/A} = -iq\Omega \left\{ \frac{k^2 s^2}{\tau} \left[ k\left( g_{ne} + g_{Te} + \frac{10}{3} \right) - \Omega \tau d_e^2 - \frac{10}{3} \frac{(g_{ne} + g_{Te})}{\Omega} k^2 \right] + \left( \frac{1}{2} - s \right) \left[ 2k\left( \frac{1}{\tau} + \frac{10}{3} + \frac{10}{3\tau} k^2 \right) - \frac{2k^2}{\tau\Omega} \left( g_{Te} - \frac{7}{3} g_{ne} + \frac{10}{3} k^2 (g_{ne} + g_{Te}) \right) \right] \right\}^{1/2} \times \left\{ \left( g_{Te} - \frac{2}{3} g_{ne} \right) \frac{k}{\tau} + \frac{10}{3} \left( 1 + \frac{1}{\tau} \right) k - \Omega \left( \frac{1}{\tau} + \frac{5}{3} \right) \right\}^{-1/2}. \tag{32}$$

In order to describe localized eigenmodes, the sign of $\sigma$ in Eq. (32) should be chosen so that the real part of $\sigma$ is positive ($\Re\sigma > 0$).

If

$$k \ll \Omega \ll k g_{Te},$$

and

$$g_{Te} \gg (1, g_{ne}),$$

one can obtain an approximation

$$\sigma = -iq\Omega ks \left[ \frac{g_{ne} + g_{Te} + 10/3}{g_{Te} - 2/3 + 10(1+\tau)k/3} \right]^{1/2}, \tag{33}$$

which is similar to Eq. (46) in Ref. 18.

A necessary condition for the application of the SBA is $|\sigma| \gg 1$, which may be verified after calculating the eigenvalue $\Omega$. In the first order of SBA, the dispersion relation for $\Omega$ is given by $B_2 = \sqrt{AC_2}$, or

$$B_0 - \frac{D^2}{4A} = B_f, \tag{34}$$

where

$$B_f = -\frac{i}{q\tau} \left\{ k^2 s^2 \left[ k\left( g_{ne} + g_{Te} + \frac{10}{3} \right) - \frac{10}{3} \frac{9 g_{ne} + g_{Te}}{\Omega} k^2 \right] + \left( \frac{1}{2} - s \right) \left[ 2k\left( 1 + \frac{10\tau}{3} + \frac{10k^2}{3} \right) - \frac{2k^2}{\Omega} \left( g_{Te} - \frac{7}{3} g_{ne} + \frac{10}{3} k^2 (g_{Te} + g_{ne}) \right) \right] \right\}^{1/2} \times \left\{ \left( g_{Te} - \frac{2}{3} g_{ne} \right) k + \frac{10}{3} (1+\tau) k - \Omega \left( 1 + \frac{5\tau}{3} \right) \right\}^{1/2}$$
$$\approx -\frac{isk^2}{q\tau} \left\{ \left( g_{Te} + g_{ne} + \frac{10}{3} \right) \times \left( g_{Te} - \frac{2}{3} g_{ne} + \frac{10}{3} (1+\tau) \right) \right\}^{1/2}. \tag{35}$$

The solution of Eq. (34) may be found using a perturbative method. In the zeroth approximation, one obtains the solution

$$\Omega = \Omega_0 = \Omega_{r0} + i\Gamma_0$$

of the local dispersion relation

$$B_0 = B(g(\theta = 0)) = 0, \tag{36}$$

which was analyzed in Refs. 17 and 18.

For reactively unstable ETG modes, considering $D^2/(4A)$ and $B_f$ as corrections in the first order SBA, a dispersion relation is obtained in the form

$$(1 + k^2 d_e^2)[(\Omega - \Omega_{r0})^2 + \Gamma_0^2] = \frac{D^2}{4A}(\Omega_0) + B_f(\Omega_0). \tag{37}$$

Separating real and imaginary parts of the right-hand side of Eq. (37), one obtains the equation

$$(\Omega - \Omega_{r0})^2 + \Gamma_0^2 = R + iI. \tag{38}$$

The real and imaginary parts of the solutions of this equation for unstable modes are

$$\Re\Omega = \Omega_{r0} + \left\{ \frac{\sqrt{I^2 + (\Gamma_0^2 - R)^2} - (\Gamma_0^2 - R)}{2} \right\}^{1/2} \tag{39}$$

and

$$\Im\Omega = \left\{ \frac{(\Gamma_0^2 - R) + \sqrt{I^2 + (\Gamma_0^2 - R)^2}}{2} \right\}^{1/2}. \tag{40}$$

The term $I$ in Eq. (38) usually corresponds to the influence of the magnetic shear, resulting in shear damping for a dissipative drift wave or amplification for a reactive drift wave.

In the case of $(\Gamma_0^2 - R)^2 \gg I^2$, Eqs. (40) and (39) can be simplified to

$$\Im\Omega = (\Gamma_0^2 - R)^{1/2} \left[ 1 + \frac{1}{8} \frac{I^2}{(\Gamma_0^2 - R)^2} \right] \tag{41}$$

and

$$\Re\Omega \approx \frac{I}{2(\Gamma_0^2 - R)^{1/2}} + \Omega_{r0}. \tag{42}$$

This corresponds to the toroidal branch of the ETG mode.

In the opposite case, with $I^2 \gg (\Gamma_0^2 - R)^2$, one has the slab-like branch of the mode

$$\Im\Omega \approx \Re(\Omega - \Omega_{r0}) \approx I/\sqrt{2}. \tag{43}$$

The transition from the toroidal branch to the slab-like branch occurs at $s \sim 2q$, where $s \ll 2q$ corresponds to the toroidal branch.[26,17,18]

Some approximations for the ETG mode can be made well above the threshold where

$$\Gamma_0^2 = \frac{2\tau k^2 g_{Te}}{1 + k^2 d_e^2} \approx 2\tau k^2 g_{Te} \gg \Omega_{r0}^2. \tag{44}$$

In this limit, the condition can be found where the strong ballooning approximation is valid. Using the approximation

$$|\sigma| \approx qks\Gamma_0 \approx k^2 qs\sqrt{2\tau g_{Te}}, \tag{45}$$

for $\sigma^2 \gg 1$, it is determined that the following inequality must be satisfied:

$$k^2 > [sq\sqrt{2\tau g_{Te}}]^{-1}. \tag{46}$$

Note that for the toroidal branch, where $\sigma/2q \ll 1$, the correction to the growth rate $\Gamma_0$ caused by the $B_f$ term must be small

$$|B_f/\Gamma_0^2| \sim \frac{s}{2\tau^2 q} \ll 1.$$

If the two terms on the right-hand side of Eq. (37) are compared, it can be shown that

$$\left| B_f \bigg/ \frac{D^2}{4A} \right| \sim \left| \frac{4\sigma A^2}{D^2} \right| \gg 1,$$

where

$$D = \frac{ks}{qc_e\Omega} \frac{dv_{0\parallel}}{dy} = \frac{1}{q\Omega\Omega_e} \frac{dv_{0\parallel}}{d\rho},$$

and

$$|A| \sim \frac{kg_{Te}}{|\Omega| q^2}.$$

Thus, the growth rate of the ETG mode changes, as a function of sheared toroidal rotation $(dv_{0\parallel}/d\rho)$, even more weakly than as a function of magnetic shear. In the following, terms of order $D^2$ are neglected for simplicity.

Finally, consider the spatial structure of the ETG mode as it is given by the first order SBA, where $G_0(y)$ is constant. In this approximation, the mode $G(y,\theta)$ in Eq. (21) involves many poloidal harmonics:

$$G(y,\theta) = \sum_m e^{-im\theta} \sqrt{\frac{2\pi}{\sigma}} f_0 \exp\left( -\frac{(y-m)^2}{2\sigma} \right). \tag{47}$$

The sum over $m$ may be replaced by an integral multiplied by $1/2\pi$, which yields

$$G(y,\theta) = f_0 \exp[-\sigma\theta^2/2 + iy\theta - y^2/\sigma]. \tag{48}$$

After integration, the eigenmode "loses" its periodicity. However, Eq. (48) is valid because the mode is strongly localized near $\theta = 0$.

Thus, the ETG mode in SBA has a width in radial direction: $\overline{y} = \sqrt{\sigma}$ or for the dimensional radial variable,

$$\overline{\rho} = \sqrt{\sigma}/k_\theta s. \tag{49}$$

Equation (49) has been used in Refs. 17 and 18 for the characteristic width of the ETG mode. This width is of the order of the electron Larmor radius, and it is too small to explain anomalous electron heat conductivity. A correct approximation for the mode radial width can be obtained in the second order of SBA, which takes into account spatial variation of the plasma parameters—in particular, strongly nonuniform backgrounds flows.

### B. The second order strong ballooning approximation

The local dispersion relation $\Omega(\eta_k, y)$ obtained in the first order SBA may be written in a form where the nonuniform background flow is taken into account:

$$B_1 - E(\cos\eta_k - \alpha_m \sin^2\eta_k) - \frac{E_1^2}{4C_1} = B_f. \tag{50}$$

Here, in Eqs. (24)–(27) for $B_1$, $E$, $E_1$, and $C_1$, $\Omega$ is replaced by $\Omega - 2Ky - Py^2$. As it has been previously observed, under the condition given by Eq. (46), the function $f_1$ is strongly localized at $\eta = \eta_k = 0$, which corresponds to a strong localization of the mode $G(y,\theta)$, Eq. (21), in the poloidal direction at $\theta = 0$. Because of this, Eq. (50) can be expanded near $\eta_k = 0$ to obtain

$$B_0 + E_2\eta_k^2 = B_f, \tag{51}$$

where

$$B_0 = B_1 - E = [(\Omega - \Omega_{r0} - 2Ky - Py^2)^2 + \Gamma_0^2](1 + k^2 d_e^2) \tag{52}$$

and

$$E_2 = \left( \frac{1}{2} + \alpha_m + \frac{E(1+s-2\alpha_m)^2}{4[C + E(1/2 - s + \alpha_m)]} \right) E. \tag{53}$$

The term $E_2\eta_k^2$ produces a small correction to the eigenvalue because $|E_2| \ll \Gamma_0^2$ and $\eta_k^2 \ll 1$. Using this fact, the local dispersion relation, Eq. (51), can be rewritten as

$$\begin{aligned} \Omega - \Omega_{r0} - 2Ky - Py^2 &\approx i\Gamma_0 \sqrt{1 - \frac{B_f - E_2\eta_k^2}{\Gamma_0^2(1+k^2d_e^2)}} \\ &\approx i\Gamma_0 \left[ 1 - \frac{B_f}{2\Gamma_0^2(1+k^2d_e^2)} - \frac{B_f^2}{8\Gamma_0^4(1+k^2d_e^2)^2} \right. \\ &\quad \left. + \frac{E_2\eta_k^2}{2\Gamma_0^2(1+k^2d_e^2)} \right]. \end{aligned} \tag{54}$$

In the case with $K = P = E_2 = 0$, the result in Eq. (41) is recovered for the growth rate, using $B_f \approx iI$.

To obtain the differential equation that describes the mode structure, it is sufficient to replace $\eta_k^2$ in Eq. (54) with the operator $-\partial^2/\partial y^2$ or, equivalently, to replace $y$ on the left-hand side of Eq. (54) with the operator $-i\partial/\partial\eta_k$ (see Refs. 20 and 22, for example).

The coefficients $E_2$, and $B_f$ will be considered as depending on $\Omega_0 \approx i\Gamma_0$ (i.e., $\Omega$ is replaced by $i\Gamma_0$). Then, the eigenvalue problem in the second order SBA is reduced to the equation,

$$iL\frac{d^2G_0}{dy^2} + (\Omega - \Omega_0 - i\Gamma_1 + K^2/P)G_0 + P(y+K/P)^2 G_0 = 0, \tag{55}$$

where

$$L \equiv \frac{E_2}{2\Gamma_0(1+k^2d_e^2)} \approx \frac{E(\alpha_m+1/2)}{2\Gamma_0(1+k^2d_e^2)} \approx -\frac{k}{\tau^{3/2}}\frac{\sqrt{g_{Te}}(\alpha_m+1/2)}{\sqrt{2}}, \tag{56}$$

and

$$\Gamma_1 = \Gamma_0\left[1 - \frac{B_f}{2\Gamma_0^2(1+k^2d_e^2)} - \frac{B_f^2}{8\Gamma_0^4(1+k^2d_e^2)^2}\right] \approx \Gamma_0\left[1 - \frac{\sqrt{A[C+E(1/2-s)]}}{2\Gamma_0^2(1+k^2d_e^2)}\right].$$

It is assumed in Eq. (55) that the radial derivative of poloidal flow shear rate is essential. Equation (55) is not applicable to the case when $P$ is very small or zero. These asymptotic cases were considered in Refs. 11 and 12.

The background eigenfunction of Eq. (55) is

$$G_0(y) = \exp\left[-\frac{\kappa}{2}(y+K/P)^2\right], \tag{57}$$

where

$$\kappa = \sqrt{iP/L}. \tag{58}$$

The sign in this expression for $\kappa$ should be chosen such that $\Re\kappa > 0$. The corresponding eigenvalue is

$$\Omega = \Omega_{r0} + i\Gamma_1 - K^2/P - i\frac{\sqrt{PL}}{2\sqrt{2}\Gamma_0(1+k^2d_e^2)} \approx \Omega_{r0} + i\Gamma_0 - \frac{K^2}{P} - \frac{i\sqrt{A(C+E(1/2-s))}}{2\Gamma_0(1+k^2d_e^2)} - \frac{i\sqrt{PL}}{2\sqrt{2}\Gamma_0(1+k^2d_e^2)}. \tag{59}$$

Let us estimate the effect of the nonuniform background flow, which is described by the parameter $P$ defined by Eq. (20). According to Fig. 2 in Ref. 1 and Fig. 2 in Ref. 2, one can approximate

$$\left(\frac{a}{\Omega_e}\frac{d\omega_E}{dr}\right)^{1/2} \approx \frac{1}{600},$$

where $a$ is the minor radius of the plasma. Using this approximation, one can show that the following inequalities are valid:

$$\left|\frac{\sqrt{PL}}{\Gamma_0}\right| \approx \left|\frac{\sqrt{PE}}{\Gamma_0}\right|\sqrt{1+2\alpha_m} \approx \frac{1}{2s\tau}\left(\frac{Rd\omega_E/dr}{\Omega_e k_\theta \rho_e}\right)^{1/2}(1+2\alpha_m)^{1/2} \ll 1,$$

and

$$\left|\frac{K^2}{P\Gamma_0}\right| \approx \frac{R^2\omega_E^2}{2c_e(2\tau g_{Te}\rho_e|d\omega_E/dr|)} \ll 1.$$

Thus, the influence of nonuniform background flows on the growth rate of the ETG mode well above threshold is negligible. However, not far from marginal stability, the last term in Eq. (48) $[\sim -i\sqrt{PL}/\Gamma_0]$ produces a stabilizing effect, which becomes stronger in plasmas with a large Shafranov shift, especially when combined with the effect of small magnetic shear $s$. Background flows determine the mode structure in the radial direction. In the second order SBA, the spatial mode structure $G(y,\theta)$, Eq. (21), is given by

$$G(y,\theta) = \exp\left[-\frac{\sigma\theta^2}{2} + iy\theta - \frac{\kappa}{2}(y+K/P)^2\right]. \tag{60}$$

From Eq. (60), one obtains the square of the characteristic size of the mode in the radial direction,

$$\overline{y^2} = k_\theta^2 s^2 \overline{\rho^2} = |\kappa|^{-1} = \left|\frac{L}{P}\right|^{1/2},$$

where $L$ is given by Eq. (56) and $P$ is given by Eq. (20).

Well above the threshold, $|E|$ and $|C|$ can be approximated

$$|E| \approx 2k^2 g_{Te}/\tau, \tag{61}$$

and

$$|C| \approx k^4 s^2 g_{Te}^{3/2}\tau^{-1/2}. \tag{62}$$

Thus

$$\left|\frac{E}{C}\right| \approx \frac{2}{k^2s^2\sqrt{\tau g_{Te}}} = \frac{2^{3/2}q}{s\sigma} < 1 \tag{63}$$

if $s/(2^{3/2}q) > \sigma^{-1}$. In this case, $L \approx E(\frac{1}{2}+\alpha_m)$.

Finally the following estimate for the square of the characteristic mode width is obtained

$$\frac{\overline{\rho^2}}{\rho_e^2} = \frac{\sqrt{g_{Te}}}{s\sqrt{k\tau}}\left(\frac{\Omega_e}{Rd\omega_E/dr}\right)^{1/2} \approx \frac{600}{s}\left(\frac{ag_{Te}}{Rk\tau}\right)^{1/2}, \tag{64}$$

where $a$ and $R$ are minor and major radii of the torus. Thus, the characteristic width of the mode is much larger than the electron Larmor radius, $\overline{\rho} \gg \rho_e$, for typical parameters such as $\sqrt{ag_{Te}/R} \geq 2$, $k\tau \sim 1$, and $s \leq 1/2$.

The width of the mode in terms of the dimensionless variable $\overline{y}$ is large:

$$\overline{y} \approx ks(\overline{\rho}/\rho_e) \gg 1,$$

which means that each mode extends through many neighboring rational surfaces. The shift of the mode localization from the point $y=0$ is given by $K/P$, which is of order

$$\left|\frac{K}{P}\right| \sim \left|k_\theta \frac{\omega_E}{d\omega_E/d\rho}\right| \gg 1,$$

which is much smaller than the mode characteristic width $\overline{y}$.

The radial width of the ETG mode given in Eq. (64) goes to infinity if the magnetic shear, $s$, goes to zero. However, the validity of this expression is restricted to values of magnetic shear that are not very small. The SBA method can be used only if the requirement in Eq. (46) is fulfilled. In order to extend the region of applicability to smaller values of magnetic shear, the direct method is used in the next section.

## IV. DIRECT METHOD

In this section the starting point is again the basic 2D equation, Eq. (14), which is now rewritten as follows:

$$\left\{\frac{\partial^2}{\partial y^2} - \Sigma\left(\frac{\partial}{\partial\theta} + iy\right)^2 + \alpha[g(\theta)-1] + 2iK_0\left(\frac{\partial}{\partial\theta} + iy\right) - \Gamma\right\}G = 0, \tag{65}$$

where

$$\Sigma \equiv -\frac{A}{C}, \quad K_0 \equiv \frac{D}{2C}, \quad \alpha \equiv \frac{E}{C},$$

$$\Gamma = \frac{[(\Omega-\Omega_{r0}-2Ky-Py^2)^2+\Gamma_0^2](1+k^2d_e^2)}{C} \approx \frac{2i\Gamma_0[\Omega-\Omega_{r0}-i\Gamma_0-2Ky-Py^2](1+k^2d_e^2)}{C} = \lambda + 2\psi y + py^2, \tag{66}$$

and $g(\theta)$ is given by Eq. (18), where the Shafranov shift is now taken to be zero, that is, $\alpha_m = 0$, for simplicity. The values of $A$, $C$, $D$, and $E$ are given by Eqs. (15), (16), (17), (23) with $\Omega$ replaced by $\Omega - 2Ky - Py^2$. In Eq. (66), the notation is

$$\lambda = \frac{2i\Gamma_0(\Omega-\Omega_{r0}-i\Gamma_0)(1+k^2d_e^2)}{C}, \tag{67}$$

$$\psi = -\frac{2i\Gamma_0 K(1+k^2d_e^2)}{C},$$

and

$$p = -\frac{2i\Gamma_0 P(1+k^2d_e^2)}{C}.$$

The expression for $\Gamma$ is simplified by taking into account the consideration presented in the previous section. In particular, under conditions well above the threshold, the growth rate of the ETG mode is not changed very much by nonuniform plasma rotation or by the term that describes the effect of magnetic curvature ($\sim\alpha$). This property is qualitatively different from that associated with the ITG mode, which can be stabilized by a nonuniform plasma rotation, as was shown in Ref. 10.

The procedure described in Ref. 12 is followed for the case with $p \neq 0$. The Fourier expansion of the eigenfunction $G$,

$$G(y,\theta) = \sum_m C_m G_m(y) e^{im\theta}, \tag{68}$$

yields a set of equations for the coefficients $C_m$ and the functions $G_m(y)$,

$$C_m L_m G_m - \frac{\alpha}{2}\left[C_{m+1}G_{m+1} + C_{m-1}G_{m-1} - 2C_m G_m - s\left(C_{m+1}\frac{\partial G_{m+1}}{\partial y} - C_{m-1}\frac{\partial G_{m-1}}{\partial y}\right)\right] = 0, \tag{69}$$

where

$$L_m \equiv \frac{d^2}{dy^2} - \lambda + \Sigma(m+y)^2 - 2K_0(m+y) - 2\psi y - py^2. \tag{70}$$

With the assumption that the terms in Eq. (69) that are proportional to $\alpha$ are small, eigenfunctions $G_m(y)$ in the first approximation of the direct method are obtained as solutions of the 1D eigenvalue equations in a mixed $(m,y)$ space,

$$L_m G_m^{(n)} = \lambda_m^{(n)} G_m^{(n)}. \tag{71}$$

The background eigenfunction $(n=0)$ has the form

$$G_m^{(0)} = \exp\left[-\frac{i}{2}\sqrt{\Sigma - p}(y+y_m)^2\right], \tag{72}$$

where

$$y_m = \frac{m\Sigma - (\psi + K_0)}{\Sigma - P}.$$

The sign before $\sqrt{\Sigma - p}$ must be chosen so that $\Re(i\sqrt{\Sigma-p}) > 0$. The background eigenvalue of Eq. (71) is

$$\lambda_m^0 = -\lambda - i\sqrt{\Sigma - p} - \frac{pm^2\Sigma - 2(\psi+K_0)m\Sigma + (\psi+K_0)^2}{\Sigma - p}. \tag{73}$$

Looking for solutions of the full set of Eq. (69) as $G_m = G_m^0 + G_m^1$, where $G_m^1$ is a function that is orthogonal to $G_m^0$,

$$\int_{-\infty}^{+\infty} G_m^0 G_m^1 dy = 0,$$

and assuming

$$\int_{-\infty}^{+\infty} |G_m^1|^2 dy \ll \int_{-\infty}^{+\infty} |G_m^0|^2 dy, \tag{74}$$

equations for the coefficients $C_m$ are obtained from Eq. (68),

$$\lambda_m^0 C_m = \alpha\left(\frac{V}{2}(C_{m+1} + C_{m-1}) - C_m\right), \tag{75}$$

where the matrix element $V$ is given by

$$V=\frac{\int_{-\infty}^{+\infty}G_m^0(G_{m-1}^0+s\partial G_{m-1}^0/\partial y)dy}{\int_{-\infty}^{+\infty}|G_m^0|^2dy}$$
$$=\left(1+\frac{is\Sigma}{2\sqrt{\Sigma-p}}\right)\exp\left(-\frac{i\Sigma^2}{4(\Sigma-p)^{3/2}}\right). \quad (76)$$

As in the case with $p=0$, considered in Ref. 11, the matrix element $V$ with $p\neq 0$ does not depend on $m$.

For the ordering in Eq. (74) to be valid, it is sufficient that

$$|\alpha(V-1)|\ll\sqrt{|\Sigma|} \quad (77)$$

in the case where the coefficients $C_m$ depend only weakly on $m$, which corresponds to the case of ballooning modes. The term $\alpha C_m$ can also be included into the left-hand side of Eq. (71), as it was done in Refs. 11 and 12. This yields another sufficient condition for the validity of this method,

$$|\alpha V|\ll\sqrt{|\Sigma|}, \quad (78)$$

which is valid even when $C_m$ is strongly dependent on $m$.

Solutions of the set of equations represented by Eq. (75) were analyzed in Refs. 12 and 27. This set of equations, with $\lambda_m^0$ given by Eq. (73), is equivalent to the second order differential equation,

$$p\frac{d^2g}{du^2}-2i(\psi+K_0)\frac{dg}{du}+bg-2dg\cos u=0, \quad (79)$$

where

$$g(u)=\sum_m C_m e^{imu}$$

is the generating function and where

$$b=-\frac{\Sigma-p}{\Sigma}\left[\lambda-\alpha+i\sqrt{\Sigma-p}+\frac{(\psi+K_0)^2}{\Sigma-p}\right]$$

and

$$d=\frac{\alpha V}{2}\frac{\Sigma-p}{\Sigma}.$$

The transformation of the function $g(u)$,

$$g(u)=\mathcal{F}(u)\exp\left[i\frac{(\psi+K_0)u}{p}\right], \quad (80)$$

and the change of the independent variable to $u=2Z$ are used in order to reduce Eq. (79) to the standard Mathieu equation,

$$\frac{d^2\mathcal{F}}{dZ^2}+(a-2q_0\cos(2Z))\mathcal{F}=0, \quad (81)$$

where $a\equiv 4A_0$, $q_0\equiv -4Q$,

$$A_0\equiv\frac{b}{p}-\frac{(\psi+K_0)^2}{p^2},$$

and

$$Q\equiv-\frac{\alpha V}{2}\frac{\Sigma-p}{p\Sigma}.$$

Eigenfunctions of Eq. (79) should be $2\pi$-periodic functions of $u$. They correspond to Floquet solutions of Eq. (81) of the form

$$F(z)=F_\nu(z)=\exp(i\nu z)P(z),$$

where $\nu=-2(\psi+K_0)/p$ and $P(z)$ is a function of $z$ with period $\pi$ (since $z=u/2$). The solution of this eigenvalue problem depends on the value of parameter $q_0$, or $Q$, with variable $a$ in Eq. (81) considered as an eigenvalue.

In order to estimate the value of parameter $Q$,

$$\Omega=i\Gamma_0\approx ik\sqrt{2g_{Te}\tau}$$

is substituted into coefficients $A$, $C$, $E$, and it is assumed that $g_{Te}\gg g_{ne}$. Then, the coefficients $A$, $C$, and $E$ can be written

$$A\approx-\frac{ig_{Te}^{1/2}}{2\tau^{3/2}q^2},\quad C\approx-i\sqrt{\frac{2}{\tau}}k^4s^2g_{Te}^{3/2},\quad E=-2k^2\frac{g_{Te}}{\tau},$$

and

$$\Sigma=-\frac{1}{\sigma^2}\approx-\left(\frac{s_{\text{crit}}}{k^2s}\right)^2,$$

where $s_{\text{crit}}\equiv|q(2\tau g_{Te})^{1/2}|^{-1}$. Thus, if the condition in Eq. (46) is valid, it follows that

$$|\Sigma|\ll 1 \quad (82)$$

and if $s/(2q)>|\Sigma|^{1/2}$, then

$$|\alpha|=\left|\frac{E}{C}\right|\approx\frac{2q|\Sigma|^{1/2}}{s}<1. \quad (83)$$

Note that for typical values of tokamak parameters,

$$\left|\frac{p}{\Sigma}\right|=\left|\frac{2\Gamma_0P}{C\Sigma}\right|=\left|\frac{R}{ks^2A\Omega_e}\frac{d\omega_E}{dr}\right|=\left|\frac{R\tau^{3/2}q^2}{\Omega_ek^2s^2g_{Te}^{1/2}}\frac{d\omega_E}{dr}\right|\ll 1. \quad (84)$$

Now the matrix element $V$ can be estimated,

$$V=\left[1+\frac{is\Sigma}{2\sqrt{\Sigma-p}}\right]\exp\left\{-\frac{i\Sigma^2}{4(\Sigma-p)^{3/2}}\right\}$$
$$\approx\left[1+\frac{is\Sigma^{1/2}}{2}\right]\exp\left(-\frac{i\Sigma^{1/2}}{4}\right)\approx\left[1+\frac{s_{\text{crit}}}{2k^2}\right]\exp\left\{-\frac{s_{\text{crit}}}{4k^2s}\right\}. \quad (85)$$

If $|\Sigma|\ll 1$, which is the case when SBA holds, then

$$V\approx 1+\frac{i}{2}\left(s-\frac{1}{2}\right)\Sigma^{1/2}, \quad (86)$$

so that $|V-1|\ll 1$, and

$$|\alpha(V-1)|<|\Sigma|^{1/2}. \quad (87)$$

The condition in Eq. (87) coincides with the sufficient condition given in Eq. (77) for the applicability of the DM if $|\Sigma|\ll 1$. Thus, if SBA is applicable, the same is true for the DM.

In the case $|\Sigma|\gg 1$, where SBA is violated, the matrix element $V$ becomes exponentially small,

$$V\approx\left(1+\frac{s_{\text{crit}}}{2k^2}\right)\exp\left\{-\frac{s_{\text{crit}}}{4k^2s}\right\}\ll 1 \quad (88)$$

for

$$|\Sigma^{1/2}| \approx \frac{s_{\rm crit}}{k^2 s} > 4. \quad (89)$$

Also

$$\left|\frac{\alpha V}{\Sigma^{1/2}}\right| \ll 1$$

under the condition in Eq. (88). Thus, in the case $|\Sigma| \gg 1$, DM is also valid, according to the sufficient condition in Eq. (78).

In the case $|\Sigma| \ll 1$, the parameter $Q$ is large

$$|Q| \sim \left|\frac{\alpha V}{2p}\right| \sim \left|\frac{\alpha}{2p}\right| \sim \left|\frac{\Omega_e}{k g_{Te}^{1/2} \frac{d\omega_E}{dr} R}\right| \gg 1$$

for typical tokamak plasmas. However, in the case $|\Sigma| \gg 1$, where the magnetic shear satisfies the condition

$$s \ll \frac{s_{\rm crit}}{4k^2},$$

it is possible that $|Q| \ll 1$.

Let us now compare the influence of the poloidal and toroidal shear plasma flows,

$$\left|\frac{\psi}{K_0}\right| \approx \left|\frac{4\Gamma_0 K}{D}\right| \approx \left|\frac{2\Gamma_0^2 q R}{s\rho_e} \frac{\omega_E}{\frac{dV_{0\parallel}}{dr}}\right| \approx \left|\frac{2\tau k^2 g_{Te} q}{s} \frac{R}{\rho} \frac{\omega_E}{\frac{dV_{0\parallel}}{dr}}\right|,$$

so that $|\psi| \gg |K_0|$ if $|dV_{0\theta}/dr|$ and $|dV_{0\parallel}/dr|$ are of the same order of magnitude.

In the next two subsections, the eigenvalue problem is examined by DM in two limiting cases, $|Q| \gg 1$ and $|Q| \ll 1$.

### A. Case A: $|Q| \gg 1$, $k^2 s \gg s_{\rm crit}$

In this case, the eigenvalue $\Omega$ of the problem is given by the dispersion relation

$$A_0 = -2Q + Q^{1/2} \quad (90)$$

or

$$\lambda = \alpha(1-V) - i\sqrt{\Sigma - p} + \frac{\Sigma + p}{\Sigma - p} \frac{(\psi + K_0)^2}{p} - i\left[\frac{\alpha p V \Sigma}{2(\Sigma - p)}\right]^{1/2}$$

$$\approx \alpha(1-V) - i\Sigma^{1/2} + \frac{\psi^2}{p} - i\left(\frac{\alpha p V}{2}\right)^{1/2}. \quad (91)$$

In the last approximation, account was taken of the fact that $|p| \ll |\Sigma|$, $|\psi| \ll |K_0|$.

The expression for $\lambda$ in Eq. (67), the expression for $V$ in Eq. (86), and the expressions for $\alpha$, $\psi$, $\rho$ in terms of $A$, $B$, $C$, $P$, and $K$ are substituted into Eq. (91) to yield

$$\Omega = \Omega_{r0} + i\Gamma_0 - \frac{i\sqrt{AC}}{2\Gamma_0(1 + k^2 d_e^2)} - \frac{iE(\frac{1}{2} - s)}{4\Gamma_0(1 + k^2 d_e^2)}$$

$$- \frac{K^2}{P} - \frac{i\sqrt{PE}}{\Gamma_0(1 + k^2 d_e^2)}. \quad (92)$$

Under the condition of Eq. (83), $|E| \ll |C|$, the term proportional to $\sqrt{A[C + E(\frac{1}{2} - s)]}$ in Eq. (59), which was obtained in SBA, can be expanded as a series in $|E/C|$. It then follows that the results for the eigenvalue obtained by SBA and DM coincide. This is natural because both methods are applicable for case A.

The spatial structure of the mode given by Eq. (65) with $K_0 = 0$ was described in detail in Ref. 12. In the limit $|p| \ll |\Sigma|$ and $Q \gg 1$, this spatial structure coincides with the one described by the SBA method, given by Eq. (60). In the next subsection the limit $Q \ll 1$, where SBA is not valid, is considered.

### B. Case B: $|Q| \ll 1$, $k^2 s \ll s_{\rm crit}$

If $|Q| \ll 1$, a solution of Eq. (79) can be obtained as an expansion,

$$g(u) = 1 + C_1 e^{iu} + C_{-1} e^{-iu} + C_2 e^{i2u} + C_{-2} e^{-i2u} + \cdots \quad (93)$$

since the coefficients $C_{\pm l}$ decay rapidly with $l$ in this case.

Only the first three terms in the series on the right-hand side of Eq. (93) are retained and substituted into Eq. (79). Terms proportional to $e^{\pm iu}$ are equated, yielding

$$C_1 \approx \frac{d}{2\psi - p}, \quad C_{-1} \approx -\frac{d}{2\psi + p}, \quad (94)$$

where $d = \alpha V/2 \approx Qp$ and $|d| \ll (|\psi|, |p|)$. The corresponding eigenvalue is determined by the dispersion relation,

$$b = -\frac{2d^2 p}{4\psi^2 - p^2}. \quad (95)$$

In the case with $p = 0$, it is found that

$$g(u) \approx 1 + \frac{id}{2\psi} e^{iu} - \frac{id}{2\psi} e^{-iu} \approx \exp\left[-i\frac{d}{\psi} \sin u\right]. \quad (96)$$

The solution for Eq. (96) was found in Ref. 11 for the case of a velocity profile that is linear in the minor radius. In this limiting case, Eq. (79) reduces to a first order differential equation. The periodicity requirement in poloidal angle of the modes yields one restriction on the frequency, but it is not enough for a full determination of its real and imaginary parts. Also, the position of the mode center, $r_0$, remains uncertain. For example, for dissipative drift electron modes, this restriction can define the growth rates, but leaves the real frequency dependent on the arbitrary value of $r_0$.[11,22] This difficulty does not arise when accounting for the nonlinearity of the background velocity profile, or when

$$\frac{d\omega_E}{d\rho} \sim p \neq 0.$$

The dispersion relation (95) determines both the real and the imaginary parts of the frequency of the background mode. With the definitions of $b$ and $d$ given after Eq. (79), Eq. (95) can be rewritten in the form

$$2i\Gamma_0(\Omega-\Omega_{r0}-i\Gamma_0)(1+k^2d_e^2)$$
$$=E-i\sqrt{-AC}-\frac{\psi^2C^2}{A}+\frac{E^2V^2}{2(\psi^2-p^2)C^2}, \quad (97)$$

with

$$V^2=(4\tau g_{Te}k^4q^2)^{-1}\exp\left(-\frac{i\Sigma^{1/2}}{2}\right), \quad (98)$$

$$\Sigma\approx-\left(\frac{s_{\mathrm{crit}}}{k^2s}\right)^2,\ \ \Re(i\Sigma^{1/2})>0.$$

The first two terms on the right-hand side of the dispersion relation in Eq. (97) do not change significantly with the growth rate of the ETG mode:

$$\left|\frac{E}{2\Gamma_0^2}\right|\sim\left|\frac{g_{Te}-\frac{7}{3}g_{ne}}{2\tau^2g_{Te}}\right|<1,\quad \left|\frac{\sqrt{AC}}{2\Gamma_0^2}\right|\sim\frac{s}{2\tau^2q}\ll 1. \quad (99)$$

Also, it can be seen that the influence of flow shear ($\omega_E\sim\psi$) on the ETG mode is usually rather small since

$$\frac{\psi^2C^2}{2A\Gamma_0^2}\sim\frac{R^2\omega_E^2\tau^{3/2}q^2}{2s^2c_e^2g_{Te}^{1/2}}<1$$

for

$$\psi^2=\frac{R^2\omega_E^2}{4s^2c_e^2}<10^{-5}$$

and $s^2g_{Te}^{1/2}/(\tau^{3/2}q^2)>10^{-5}$.

The influence of the last term in Eq. (97) is exponentially small. As will be shown below, poloidal harmonics become strongly localized with respect to the distance between mode rational surfaces and the toroidal coupling becomes negligible. Similar behavior is found for the ITG mode for very small values of magnetic shear, $s$, which was pointed out in Ref. 20. Here, a straightforward derivation of the dispersion relation will be given and the mode structure using the DM in the case under consideration, where SBA is not valid, will be determined.

The spatial structure of the mode is determined by the Fourier transform of the generation function $g(u)$, which directly yields coefficients of the original eigenfunction $G(y,\theta)$, Eq. (68):

$$G_l(y)=\int g(u)e^{-ily}du. \quad (100)$$

Using Eq. (93), the eigenfunction $G(y,\theta)$ can be written as

$$G(y,\theta)=G_0(y)+C_1G_1(y)e^{i\theta}+C_{-1}G_{-1}(y)e^{-i\theta}, \quad (101)$$

where the coefficients $C_{\pm1}$ are given by Eq. (94) and $G_l(y)$ ($l=0,\pm1$) are given by Eq. (72).

Thus, Eq. (101) can be rewritten as

$$G(y,\theta)=\exp\left[-\frac{i\sqrt{\Sigma-p}}{2}\left(y-\frac{\psi}{2}\right)^2\right]+\frac{d}{2\psi-p}$$
$$\times\exp\left[i\theta-\frac{i\sqrt{\Sigma-p}}{2}(y+y_1)^2\right]-\frac{d}{2\psi-p}$$
$$\times\exp\left[-i\theta-\frac{i\sqrt{\Sigma-p}}{2}(y-y_{-1})^2\right], \quad (102)$$

where

$$y_1=\frac{\Sigma-\psi}{\Sigma-p},\quad \text{and}\ \ y_{-1}=-\frac{\Sigma+\psi}{\Sigma-p}. \quad (103)$$

Since $|\Sigma|\gg|\psi|$ and $|\Sigma|\gg|p|$, the following result is obtained for $G(y,\theta)$:

$$G(y,\theta)=\exp\left[-\frac{i\sqrt{\Sigma}}{2}\left(y-\frac{\psi}{2}\right)^2\right]$$
$$+\frac{d}{2\psi-p}\exp\left[i\theta-\frac{i\sqrt{\Sigma}}{2}(y+1)^2\right]$$
$$-\frac{d}{2\psi+p}\exp\left[-i\theta-\frac{i\sqrt{\Sigma}(y-1)^2}{2}\right]. \quad (104)$$

Note that the condition $\Re(i\sqrt{\Sigma})>0$ applies in this equation.

Equation (104) can be simplified further by taking into account the fact that $|\psi|\gg p$ and $|\psi|\ll 1$:

$$G(y,\theta)\approx e^{-i(\sqrt{\Sigma}/2)y^2}+\frac{d}{\psi}$$
$$\times\exp\left(-\frac{i\sqrt{\Sigma}(y^2+1)}{2}\right)\sin(\theta-\sqrt{\Sigma}y). \quad (105)$$

The second term on the right-hand side of Eq. (105) may cause amplification or reduction of zonal flows.

It follows from Eq. (104) that the structure of the mode, $G(y,\theta)$, changes dramatically compared with the result in the previous case A. In this case B, the parameter $\Sigma$ is large:

$$|\Sigma|\sim\left(\frac{s_{\mathrm{crit}}}{k^2s}\right)^2\gg 1.$$

Thus the characteristic dimensionless width of each harmonic term in Eq. (104) is

$$\bar{y}\sim|\Sigma|^{-1/4}=|\sigma|^{1/2}\ll 1,$$

and the dimensional radial variable is given by Eq. (49). The characteristic width of the ETG mode in the regions of small magnetic shear $s$ becomes of the order of electron Larmor radius, $\rho_e$, and much less than in case A.

The main contribution to the mode in case B is given by the first term in Eq. (104), which describes the mode localized near the mode rational surface at $r=r_0$, where $q(r_0)=m/n$. The spatial structure expands uniformly over the poloidal direction. Other terms are exponentially small. They describe the structure of poloidal harmonics $m\pm1$, which are strongly localized near the positions of their mode rational surfaces $q=(m\pm1)/n$.

## V. CONCLUSIONS AND DISCUSSION

It was found that well above threshold $(g_{Te}\gg 1)$, the growth rate of the ETG mode changes only slightly in the presence of nonuniform plasma rotation, in the ballooning regime as well as in the opposite case, where the mode is not localized in the poloidal direction. However, the spatial mode structure is strongly influenced by background flows with a velocity profile that varies nonlinearly with minor radius. This case has been analyzed using both strong ballooning and direct methods.

The structure of the mode was investigated in the first order of the SBA in Refs. 17 and 18. It was found that the characteristic size of the mode is of order $|\sigma|^{-1/2}$ in the poloidal direction, where the parameter $\sigma$ is determined by Eq. (32). In the derivation, account is taken of the presence of impurity and superthermal ions, which lead to a redefinition of $\tau$ compared to the definition given in earlier work, as well as in the definition of the Shafranov shift $\alpha_m$, which contributes to a small reduction of the ETG growth rate. For strongly unstable ETG modes, $|\sigma|$ is given by Eq. (45).

In a general case, the strong ballooning formalism can not be applied to the toroidal drift modes in the presence of nonuniform flows. However, it has been shown by applying the second order SBA and DM to the same case (case III B and, more precisely, in the case IV A) that the SBA works if the ETG mode is strongly localized in the poloidal direction. Specifically, the ETG mode with $k_\theta^2\rho_e^2 \lesssim 1$ is strongly localized in the poloidal direction, and the SBA is valid if $|\sigma|\gg 1$ or

$$s\gg s_{\rm crit}. \tag{106}$$

For toroidal ETG modes, the magnitude of the magnetic shear is also restricted by inequality $s<1/2q$. The characteristic mode width in the radial direction[17,18] is of the order of the electron Larmor radius in the absence of background flows. With the effect of nonuniform background flows taken into account, the characteristic size of the mode in the radial direction is estimated, when the condition in Eq. (106) is fulfilled, using both the second order SBA and the DM techniques. Both methods yield similar results. Namely, the characteristic size of the mode is found to be

$$\bar{\rho}\sim (k_\theta s)^{-1}\frac{L^{1/4}}{P}, \tag{107}$$

where $P$ is given by Eq. (20), and $L$ is given by Eq. (56). Note that this size depends strongly on the parameter $P$, which is proportional to the first derivative of the flow shear rate $(P\sim d\omega_E/dr)$.

Generally speaking, the mode structure depends on many parameters that enter through $P$ and $L$, such as the safety factor, magnetic shear, gradients of the electron temperature and density, and Shafranov shift. A simplified formula, Eq. (64), is obtained for this size, assuming that the ETG mode is well above the threshold of the instability $[g_{Te}\gg (g_{ne},1)]$. The radial size of the mode is approximately 50 times larger than the electron Larmor radius. Such a radial size is rather typical for streamers, as observed in the numerical simulations reported in Ref. 8. This may explain why the electron heat transport in the plasma core is much larger than that given by a simple mixing rule $\chi_e=\gamma_0/k_\perp^2$ with $k_\perp\sim\rho_e^{-1}$. Indeed, for the ETG mode described above as a solution of the two-dimensional problem, Eq. (14), the characteristic value of $k_\perp$ is equal to $\bar{\rho}^{-1}\ll\rho_e^{-1}$ according to the estimate given by Eqs. (107) and (64). The enhanced electron thermal diffusivity may drive the temperature profile toward the marginal electron stability boundary. A critical parameter for the linear theory is not the magnetic shear itself, but the relation among the magnetic shear, the safety factor and the electron temperature characteristic length. Even when the magnetic shear is less then unity, the condition in Eq. (106) can still be satisfied and streamer-like structures can be formed. These conditions can explain some experimental observations that electron thermal transport remains anomalous in the central regions with small shear, where an ITB in the ion channel forms. For example, within the internal transport barrier the temperature gradient scale length could be 0.06 m and the major radius can be 3.0 m, so that $g_{Te}=50$. Then, if $q=2$ and $\tau=1$, $s_{\rm crit}=(2\times 50)^{-1/2}=0.1$. In this case, a magnetic shear of few tenths will be several times larger than $s_{\rm crit}$ and several times smaller than unity $(s_{\rm crit}\ll s\ll 1)$. The parameters for this example are typical for Joint European Torus (JET) discharges with the internal transport barriers [see, for example, the simulation of the JET discharge 40542 (Ref. 7)]. Also, some experimental results from DIII-D (Ref. 1) suggest that the magnetic shear is low and varies over the range $-0.16<s<0.6$ within the internal transport barrier. Case A ($|Q|\gg 1$ and $k^2s\gg s_{\rm crit}$) can be applied to the plasma within the ITBs, where the magnetic shear is of order of 0.6. Thus, the condition for case A can be satisfied when the electron transport remains strong.

The case of small magnetic shear, $s$, was considered under the condition opposite to that given by inequality in Eq. (106). In this case ($|\sigma|\ll 1$), the structure of the ETG mode changes significantly. It is extended in the poloidal direction, and the SBA method is not applicable. This case was studied in a straightforward manner using the DM. For a sufficiently small magnetic shear, toroidal coupling becomes exponentially weak, and the ETG mode becomes strongly localized in the radial direction. Under these conditions, the characteristic size of the ETG mode, $\bar{\rho}$, is given by

$$\bar{\rho}\sim (k_\theta s)^{-1}|\sigma|^{1/2},\quad s\ll (q(2\tau g_{Te})^{1/2})^{-1},\quad |\sigma|\ll 1, \tag{108}$$

which coincides with the size of the ETG mode $\bar{\rho}_m$, given in Eq. (49), which was obtained in the opposite strong ballooning limit, $\sigma\gg 1$, in the first approximation, or in the absence of nonuniform plasma rotation. Equations (49) and (108) coincide only formally, because the first formula [Eq. (49)] is valid for large values of the parameter $\sigma$, and the second formula [Eq. (108)] is valid for small values of the parameter $\sigma$. The poloidal structure of the modes is quite different in these two cases. Equation (49) describes the radial structure of the ETG mode when $|Q|\gg 1$ and $k^2s\gg s_{\rm crit}$ (case IV A) in the first order SBA, while Eq. (108) describes the radial mode structure in the opposite case when $|Q|\ll 1$ and $k^2s\ll s_{\rm crit}$ (case IV B). In the presence of nonuniform background flow in case IV A, the first order SBA is insufficient to determine the radial mode structure. In this instance, one should use the second order SBA or, alternatively, the more rigorous DM. Both methods give the same result, but use of the DM helps in understanding the difference between the radial widths obtained in the first order SBA [Eq. (49)] and the radial width obtained in the second order SBA [Eq. (64)]. Equation (49) gives the radial width of each separate poloidal mode *m* [see Eq. (72)]. The coupling between many different poloidal modes due to nonuniform flow is not taken into account at this point in our derivation. However, if $|Q|\gg 1$ and $k^2 s \gg s_{\mathrm{crit}}$ (case IV A), many poloidal modes take part in the formation of the global mode structure. This coupling was shown in detail in Ref. 12, where the sum over the mode numbers *m* in Eq. (68) was calculated. Thus, the global mode in this limit (case IV A) is composed of a large number of poloidal *m*-modes, each of which is localized around its rational magnetic surface. The distance between these rational magnetic surfaces is smaller than the spatial width of *m*-modes and, consequently, they overlap. It has been shown, in this paper, that if $|Q|\ll 1$ and $k^2 s \ll s_{\mathrm{crit}}$ (case IV B), the radial structure of the ETG mode can be described only by the direct method. In this limit of very small magnetic shear ($k^2 s \ll s_{\mathrm{crit}}$), the coupling between different poloidal modes becomes exponentially small, and only a few modes take part in the formation of the mode structure [see Eq. (102)]. As a result, the distances between different rational surfaces, which are inversely proportional to *s*, become large compared to the radial width of each drift mode. Hence, for case IV B, the mode becomes strongly localized near its rational surface and is nearly insensitive to nonuniform background flows. Thus, for case IV B, the radial width given by Eq. (108) does not depend on the nonuniform flow parameters. Consequently, the result for the radial width given in case IV B coincides with the result given in Eq. (49), where the nonuniform background flows are not taken into account. However, when nonuniform background flows are taken into account Eq. (49) is no longer valid, and the result for the radial width is given by Eq. (107). The difference in results given by Eq. (49) and Eq. (107) illustrates the role of nonuniform flows.

The characteristic radial size of the ETG mode becomes of the order of the electron Larmor radius in the limit of very small magnetic shear, when $|Q|\ll 1$ and $k^2 s \ll s_{\mathrm{crit}}$. It is expected that the electron heat transport will be strongly reduced in this case, and the formation of an electron transport barrier may result. This conclusion correlates with experimental observations of enhanced plasma confinement in regimes with low magnetic shear. Transport barriers are often formed close to the radius where safety factor *q* is minimal. It should be noticed however, that the parameter $\sigma$ depends intrinsically on the magnetic shear and the magnetic shear can become so small that $k_\theta s < |\sigma|^{1/2}$. Under these circumstances, the characteristic radial width becomes larger than the electron Larmor radius and the electron thermal transport becomes large inside the region of small magnetic shear.

Many present day tokamaks have elongated magnetic surfaces. The effect of elongation described in Ref. 28 may be take into account qualitatively if the operator $\hat{k}_\perp^2$ in Eq. (11) is written,

$$\hat{k}_\perp^2 = k_\theta^2 \left( 1 - s^2 \frac{\partial^2}{\partial y^2} \right) \tag{109}$$

is replaced by

$$\hat{k}_\perp^2 = k_\theta^2 \left( 1 - s_1^2 \frac{\partial^2}{\partial y^2} \right), \tag{110}$$

where

$$s_1^2 = (2s - 1 + (s-1)^2 \kappa^2), \tag{111}$$

and where $\kappa$ is the elongation of the magnetic surface. Note that $s_1 > s$ for $\kappa > 1$.

Accordingly, the factor $k^2 s^2/\tau$ before the square bracket in the definition of coefficient *C*, Eq. (16), should be replaced by $k^2 s_1^2/\tau$. Also, the dimensionless variable $y = k_\theta s \rho$ becomes $y = k_\theta s_1 \rho$. With this in mind, the magnetic shear parameter *s* should be replaced by $s_1$ in Eqs. (107) and (108), which describe the characteristic radial widths of the ETG mode. Consequently, the following qualitative result is obtained for both limits considered in this paper: the characteristic radial width of the ETG mode is reduced as the elongation parameter $\kappa$ is increased.

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